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Uniform Lipschitz Constants on Small Intervals

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Let f in $C[-1, 1]$ be given, and let n be a fixed nonnegative integer. For $0 < \theta \leqslant 1$ define $P_\theta(f)$ to be the polynomial of degree less than or equal to n of best uniform approximation to f on $[-\theta, \theta]$. It is well known that there exists for each such θ, a constant $\lambda_f(\theta)$ such that for all $g \in C[-\theta, \theta]$,

$$\| P_\theta(f) - P_\theta(g) \|_{[-\theta,\theta]} \leqslant \lambda_f(\theta) \| f - g \|_{[-\theta,\theta]} .$$

Sufficient conditions on f are obtained to ensure that the set $\{\lambda_f(\theta) : 0 < \theta \leqslant \delta\}$ is bounded for some $\delta > 0$. An example is given showing that $\{\lambda_f(\theta) : 0 < \theta \leqslant \delta\}$ may be bounded for some $\delta < 1$ but not for $\delta = 1$.

1. INTRODUCTION

Let $C(I)$ denote the set of continuous, real valued functions on the compact interval I, and let $M \subseteq C(I)$ be a Haar subspace of dimension n. Let $\| \cdot \|_I$ denote the uniform norm on I. For $f \in C(I)$ with best approximation $P(f)$ from M there is a constant $\lambda_f > 0$ such that for any $g \in C(I)$,

$$\| P(f) - P(g) \|_I \leqslant \lambda_f \| f - g \|_I . \tag{1.1}$$

This is Freud's well-known theorem (see [2, p. 82]). A number of recent papers [1, 3, 4, 10] have examined variants of inequality (1.1). In particular, Bartelt [1] and Cline [3] show that $\lambda = \lambda_f$ may actually be chosen independent of f if the interval I is replaced by a finite point set X. Henry and Schmidt [4] show that if Γ is a compact subset of $C(I)$ and $\Gamma \cap M = \varnothing$ then

$$\| P(f) - P(g) \|_I \leqslant \lambda_\Gamma \| f - g \|_I \tag{1.2}$$

for all $f \in \Gamma$ and $g \in C(I)$. That is, the Lipschitz constant λ_Γ in (1.2) is "uniform" over the set Γ.

Henry and Wiggins [5] utilized another kind of uniformity of Lipschitz constants to compare two local approximate solutions to an initial value

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problem. More specifically, [5] makes use of a special case of the problem now to be described.

Let $f \in C[-1, 1]$, and for $h \in C[-\theta, \theta]$, $0 < \theta \leqslant 1$, define

$$\| h \|_\theta = \sup_{-\theta \leqslant x \leqslant \theta} | h(x) |.$$

Suppose that $P_\theta(h)$ is the best approximation from the Haar subspace $M_\theta \subseteq C[-\theta, \theta]$ to $h \in C[-\theta, \theta]$. Then inequality (1.1) becomes

$$\| P_\theta(f) - P_\theta(g) \|_\theta \leqslant \lambda_f(\theta) \| f - g \|_\theta , \tag{1.3}$$

for all $g \in C[-\theta, \theta]$ where the f in (1.3) is viewed as the restriction of the original $f \in C[-1, 1]$ to the interval $[-\theta, \theta]$. Hereafter $\lambda_f(\theta)$ will designate the smallest constant for which (1.3) holds for all $g \in C[-\theta, \theta]$.

Let $M \subseteq C[-1, 1]$ be a Haar subspace, and note that the set M_θ of restrictions to $[-\theta, \theta]$ of functions in M is also a Haar subspace for each $0 < \theta \leqslant 1$. In the sequel we examine conditions on f which ensure that the set

$$\Lambda = \{\lambda_f(\theta) \mid 0 < \theta \leqslant 1\} \tag{1.4}$$

is a bounded set, with $M = \pi_n$, the set of algebraic polynomials of degree n or less.

The above description suggests that this paper might well be characterized as an additional study of Chebyshev approximations on small intervals. Other investigations of Chebyshev approximations on small intervals include those of Maehly and Witzgall [7], Meinardus [8], and Nitsche [9].

2. An Example on Small Intervals

In this section we construct an example which demonstrates that in general, the set (1.4) is not bounded. This result is somewhat surprising, in that one might expect the shrinking interval process to at least produce a uniform Lipschitz constant for all θ sufficiently small. That is, even if (1.4) is an unbounded set, one might expect that there exists a sufficiently small $\delta > 0$ such that

$$\{\lambda_f(\theta) \mid 0 < \theta \leqslant \delta\} \tag{2.1}$$

is a bounded set. We propose to construct an $f \in C[-1, 1]$ and corresponding sequences $\{z_k\}$ and $\{f_k\}$ with $z_k \to 0$ and $f_k \in C[-z_k, z_k]$ such that

$$\lim_{k\to\infty} \frac{\| P_{z_k}(f) - P_{z_k}(f_k)\|_{z_k}}{\| f - f_k \|_{z_k}} = +\infty. \tag{2.2}$$

Thus (2.1) is in general not a bounded set for any $\delta > 0$. Throughout this section all approximates are from π_1.

EXAMPLE 1. *Define for real numbers $u < v$ the function*

$$h_{uv}(x) = (x - u)^4 (x - v)^4. \tag{2.3}$$

Note that h_{uv} attains its maximum value on $[u, v]$ at the midpoint of $[u, v]$ and that

$$\| h_{uv} \|_{[u,v]} = \left(\frac{u - v}{2}\right)^8. \tag{2.4}$$

Let real numbers $c > 0$, $a > 0$, and $-1 \leqslant t < \sigma < \eta < \alpha < \beta < \gamma < \delta < \tau < z \leqslant 1$ be given, where $\beta - \alpha = \gamma - \beta = \delta - \gamma = \epsilon$. Define s, $g \in C[-1, 1]$ as follows:

$$\begin{aligned} s(x) &= ch_{\alpha\beta}(x) && \text{for } \alpha \leqslant x \leqslant \beta \\ &= -ch_{\beta\gamma}(x) && \text{for } \beta \leqslant x \leqslant \gamma \\ &= ch_{\gamma\delta}(x) && \text{for } \gamma \leqslant x \leqslant \delta \\ &\;\;\; 0 && \text{elsewhere;} \end{aligned} \tag{2.5}$$

$$\begin{aligned} g(x) &= s(x) + a(x - \sigma) && \text{for } \alpha \leqslant x \leqslant \delta \\ &= 0 && \text{on the complement of } [\eta, \tau] \\ &= \text{linear on } [\eta, \alpha] \text{ and on } [\delta, \tau]. \end{aligned} \tag{2.6}$$

With this notation we prove the following lemma.

LEMMA 1. *Let $a > 0$ satisfy*

$$\max[a(z - \sigma), a(\sigma - t)] \leqslant c(\epsilon/2)^8. \tag{2.7}$$

Then the polynomials of best approximation from π_1 to s and g, respectively, on $[t, z]$ are given by

$$p(x) = 0 \qquad \textit{and} \qquad q(x) = a(x - \sigma). \tag{2.8}$$

Proof. The first assertion is a simple consequence of the Chebyshev alternation theorem [2, p. 75]. The alternation theorem also implies that q is the best approximation to g on $[\alpha, \delta]$ with $\| g - q \|_{[\alpha,\delta]} = c(\epsilon/2)^8$. Now (2.7) implies that

$$\| g - q \|_{[t,z]} = c(\epsilon/2)^8.$$

Hence the alternation theorem implies that q is the best approximation from π_1 to g on $[t, z]$, concluding the proof of the second assertion.

Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers with $3 \leqslant n_1$ and $n_k^2 < n_{k+1}$, $k = 1, 2, \ldots$. For $k = 1, 2, \ldots$, define

$$\epsilon_k = \tfrac{1}{2}\left(\frac{1}{n_k^2} - \frac{1}{n_{k+1}}\right). \tag{2.9}$$

Now define real numbers $-1 < t_k < \sigma_k < \eta_k < \alpha_k < \beta_k < \gamma_k < \delta_k < \tau_k < z_k < 1$, $k = 1, 2,\ldots$, as follows:

$$
\begin{aligned}
t_k &= -1/n_k\,;\\
\sigma_k &= 0;\\
\eta_k &= (1/n_k^2) - 2\epsilon_k = 1/n_{k+1}\,;\\
\alpha_k &= (1/n_k^2) - \tfrac{3}{2}\epsilon_k\,;\\
\beta_k &= (1/n_k^2) - (\epsilon_k/2);\\
\gamma_k &= (1/n_k^2) + (\epsilon_k/2);\\
\delta_k &= (1/n_k^2) + \tfrac{3}{2}\epsilon_k\,;\\
\tau_k &= (1/n_k^2) + 2\epsilon_k < 1/n_k\,;\\
z_k &= 1/n_k\,.
\end{aligned}
\tag{2.10}
$$

Let $c_k > 0$ and $a_k > 0$, $k = 1, 2,\ldots$ be positive integers to be determined later.

We now define for $k = 1, 2,\ldots, s_k(x)$ and $g_k(x)$ to be as in (2.5) and (2.6), respectively, with c, a, t, σ, η, α, β, γ, δ, τ, and z replaced by c_k, a_k, t_k, σ_k, η_k, α_k, β_k, γ_k, δ_k, τ_k, and z_k, respectively.

Now let

$$f(x) = \sum_{j=1}^{\infty} s_j(x) \tag{2.11}$$

for $-1 \leqslant x \leqslant 1$, and

$$f_k(x) = g_k(x) + \sum_{j=k+1}^{\infty} s_j(x) \tag{2.12}$$

for $-1 \leqslant x \leqslant 1$ and $k = 1, 2,\ldots.$

Let us observe that the restriction of f to $[n_{j+1}^{-1}, n_j^{-1}]$ is $s_j(x)$ and this is also the restriction of f_k for $j \geqslant k+1$. In turn, $f_k(x) = g_k(x)$ for $x \in [n_{k+1}^{-1}, n_k^{-1}]$ and $f_k(x)$ vanishes for $x > n_k^{-1}$.

Define

$$a_k = c_k(\epsilon_k/2)^8 n_k\,, \tag{2.13}$$

where the c_k are chosen recursively as follows: Choose $c_1 = 1$ and c_{k+1} small enough to ensure that $c_{k+1} < c_k$ and

$$c_{k+1}\left(\frac{\epsilon_{k+1}}{2}\right)^8 + a_k\eta_k \leqslant c_k\left(\frac{\epsilon_k}{2}\right)^8. \tag{2.14}$$

Lemma 1 guarantees that the polynomial of best approximation from π_1 to s_k on $[t_k, z_k] = [-z_k, z_k]$ is $p_k(x) \equiv 0$. Since $c_{k+1} < c_k$, the construction of f implies that the polynomial of best approximation from π_1 to f on $[-z_k, z_k]$ is also $p_k(x) \equiv 0$.

Now a_k satisfies (2.13), and consequently Lemma 1 implies that the best approximation from π_1 to g_k on $[-z_k, z_k]$ is $q_k(x) = a_k x$. Inequality (2.14) and the construction of f_k then guarantee that q_k is the best approximation from π_1 to f_k on $[-z_k, z_k]$.

One can easily verify that

$$\|f - f_k\|_{z_k} = a_k\delta_k \tag{2.15}$$

and that

$$\|p_k - q_k\|_{z_k} = a_k z_k . \tag{2.16}$$

Equations (2.15), (2.16), and (2.9) finally imply that

$$\lim_{k\to\infty} \frac{\|p_k - q_k\|_{z_k}}{\|f - f_k\|_{z_k}} = +\infty ,$$

and consequently (2.2) is established for the f and f_k of (2.11) and (2.12).

In concluding Section 2 we note that c_k, $k = 1, 2, \ldots$, can be chosen sufficiently small to ensure that $f \in C^2[-1, 1]$.

3. Sufficient Conditions for Uniform Lipschitz Constants

The following theorem is the principal result of this section. We assume throughout this section that n is given and that all approximates are from π_n unless stated otherwise.

THEOREM 1. *Let $f \in C[-1, 1]$, and suppose there exists a number δ, $0 < \delta \leq 1$ such that $f \in C^{n+1}[-\delta, \delta]$, and such that $f^{(n+1)}$ does not change sign on $[-\delta, 0)$ and on $(0, \delta]$. Furthermore, suppose there exist positive constants m and M, and a polynomial $p \in \pi_K$, $K \geq n$, such that*

$$0 \leq m\,|p^{(n+1)}(x)| \leq |f^{(n+1)}(x)| \leq M\,|p^{(n+1)}(x)| \tag{3.1}$$

is valid for all $x \in [-\delta, \delta]$. Then

$$\{\lambda_f(\theta) \mid 0 < \theta \leq \delta\} \tag{3.2}$$

is bounded. That is, if $P_\theta(h)$ is the best approximation from π_n to $h \in C[-\theta, \theta]$, then there exists a $\lambda_\delta > 0$ such that

$$\|P_\theta(f) - P_\theta(g)\|_\theta \leq \lambda_\delta \|f - g\|_\theta \tag{3.3}$$

for every $g \in C[-\theta, \theta]$ and for every $\theta \leq \delta$.

Remarks. We note that this theorem does not imply that the set (1.4) is bounded, but rather that a uniform Lipschitz constant exists for all θ

sufficiently small. We also note that although the f of Example 1 can be chosen in $C^2[-1, 1]$, the hypothesis (3.1) is not satisfied for any $K \geqslant 1$.

Before proceeding to the proof of Theorem 1 we state the following lemma. The result is due to Cline [3], and will be used in the proof of Theorem 1.

LEMMA 2. *Let $h \in C[-\theta, \theta]$ with $h \notin \pi_n$. Let $P \in \pi_n$ be the best approximation to h on $[-\theta, \theta]$ and for each Chebyshev alternation $E = \{t_j\}_{j=1}^{n+2}$ for $h - P$ define $q_i \in \pi_n$ by $q_i(t_j) = \operatorname{sgn}[h(t_j) - P(t_j)]$, $j = 1, 2,..., n + 2$; $j \neq i$, and $i = 1,..., n + 2$. Now let*

$$\Omega(E) = \max_{1 \leqslant i \leqslant n+2} \{\| q_i \|_\theta\}.$$

Then there exists a Chebyshev alternation E^* for $h - P$ such that

$$\lambda_h(\theta) \leqslant 2\Omega(E^*), \tag{3.4}$$

where $\lambda_h(\theta)$ is the Lipschitz constant for h on $[-\theta, \theta]$.

Proof of Theorem 1. We first note that if $f \in \pi_n$ on $[-\theta, \theta]$ for some $\theta \in (0, 1]$, then (3.1) implies that $f \in \pi_n$ on $[-\delta, \delta]$. In this case $\lambda_f(\theta) \leqslant 2$ for all $\theta \leqslant \delta$. Thus $\sup\{\lambda_f(\theta) \mid 0 < \theta \leqslant \delta\} \leqslant 2$, and (3.3) is then valid for $\lambda_\delta = 2$.

Suppose that $f \notin \pi_n$ on any interval $[-\theta, \theta] \subseteq [-1, 1]$. Let $[-\delta, \delta]$ be the interval on which hypothesis (3.1) is satisfied. By replacing f by $(-f)$, if necessary, we may assume without loss of generality either that $f^{(n+1)}(x) \geqslant 0$ for all $x \in [-\delta, \delta]$, or that $f^{(n+1)}(0) = 0$, $f^{(n+1)}(x) \geqslant 0$ on $(0, \delta]$, and $f^{(n+1)}(x) \leqslant 0$ on $[-\delta, 0)$. We presently consider this latter case. Let θ be any positive number less than or equal to δ. Then inequality (3.1) implies that

$$m \mid p^{(n+1)}(x) \mid \leqslant f^{(n+1)}(x) \leqslant M \mid p^{(n+1)}(x)| \tag{3.5}$$

for $x \in [0, \theta]$, and that

$$m \mid p^{(n+1)}(x) \mid \leqslant -f^{(n+1)}(x) \leqslant M \mid p^{(n+1)}(x)| \tag{3.6}$$

for $x \in [-\theta, 0]$. Denote by $E_n(f; a, b)$ the degree of approximation to f from the set π_n on the interval $[a, b]$. Then (3.5) and Bernstein's theorem [6, p. 38] imply that

$$E_n(mp; 0, \theta) \leqslant E_n(f, 0, \theta). \tag{3.7}$$

Since $E_n(f; 0, \theta) \leqslant E_n(f; -\theta, \theta)$, (3.7) implies that

$$E_n(mp; 0, \theta) \leqslant E_n(f; -\theta, \theta). \tag{3.8}$$

Similarly, (3.6) implies that

$$E_n(mp; -\theta, 0) \leqslant E_n(f; -\theta, \theta). \tag{3.9}$$

Let $P(\theta, p)$ be the best approximation from π_n to p on $[0, \theta]$, and let

$$e(\theta, p)(x) = p(x) - P(\theta, p)(x). \tag{3.10}$$

Then

$$\| e(\theta, p) \|_{[0,\theta]} = E_n(p; 0, \theta), \tag{3.11}$$

and

$$e^{(n+1)}(\theta, p)(x) = p^{(n+1)}(x). \tag{3.12}$$

Similarly, if $P(-\theta, p)$ is the best approximation from π_n to p on $[-\theta, 0]$, then

$$e(-\theta, p)(x) = p(x) - P(-\theta, p)(x), \tag{3.13}$$

$$\| e(-\theta, p) \|_{[-\theta,0]} = E_n(p; -\theta, 0), \tag{3.14}$$

and

$$e^{(n+1)}(-\theta, p)(x) = p^{(n+1)}(x). \tag{3.15}$$

Since (3.10) and (3.13) are both polynomials of degree at most K, Markoff's inequality [2, p. 91 and p. 94, problem 4] with (3.11) and (3.14) implies that

$$| e^{(n+1)}(\theta, p)(x) | \leqslant \frac{2^{n+1} K^{2n+2}}{\theta^{n+1}} E_n(p; 0, \theta), \tag{3.16}$$

for $x \in [0, \theta]$, and that

$$| e^{(n+1)}(-\theta, p)(x) | \leqslant \frac{2^{n+1} K^{2n+2}}{\theta^{n+1}} E_n(p; -\theta, 0) \tag{3.17}$$

for $x \in [-\theta, 0]$. Expressions (3.12), (3.15), (3.16), and (3.17) then imply

$$\| p^{(n+1)} \|_\theta \leqslant \frac{2^{n+1} K^{2n+2}}{\theta^{n+1}} \max[E_n(p; 0, \theta), E_n(p; -\theta, 0)]. \tag{3.18}$$

We note that although the constant K in (3.16), (3.17), and (3.18) does depend on n, it is independent of θ. Let $E_\theta = \{t_j\}_{j=1}^{n+2}$ be any Chebyshev alternation for

$$d(\theta, f)(x) = [f - P_\theta(f)](x) \tag{3.19}$$

where again $P_\theta(f)$ is the best approximation from π_n to f on $[-\theta, \theta]$. If $\{q_i\}_{i=1}^{n+2}$ is the set of polynomials of Lemma 2 for the Chebyshev alternation E_θ, then

$$q_i(t_j) = \frac{d(\theta, f)(t_j)}{E_n(f; -\theta, \theta)},$$

$$j = 1, 2, \dots, n+2, \qquad j \neq i, \qquad i = 1, \dots, n+2.$$

The classical remainder theorem of interpolation theory [2, p. 60] then implies that

$$\frac{d(\theta, f)(x)}{E_n(f; -\theta, \theta)} - q_i(x) = \frac{d^{(n+1)}(\theta, f)(\xi)}{E_n(f; -\theta, \theta)} \frac{w_i(x)}{(n+1)!},$$

where $w_i(x) = \prod_{j=1, j \neq i}^{n+2} (x - t_j)$, and $x, \xi \in [-\theta, \theta]$. Consequently (3.19) and (3.1) imply that

$$\left| \frac{d(\theta, f)(x)}{E_n(f; -\theta, \theta)} - q_i(x) \right| \leqslant \frac{M \mid p^{(n+1)}(\xi) \mid \mid w_i(x) \mid}{E_n(f; -\theta, \theta)(n+1)!}.$$

Therefore

$$| q_i(x) | \leqslant \frac{M \mid p^{(n+1)}(\xi) \mid \mid w_i(x) \mid}{E_n(f; -\theta, \theta)(n+1)!} + 1, \tag{3.20}$$

for $x \in [-\theta, \theta]$. Since $K > n$, (3.8) and (3.9) imply that

$$0 < \max[E_n(mp; -\theta, 0), E_n(mp; 0, \theta)] \leqslant E_n(f; -\theta, \theta).$$

But $E_n(mp; a, b) = mE_n(p; a, b)$. Consequently (3.20) implies that

$$| q_i(x) | \leqslant \frac{M \mid p^{(n+1)}(\xi) \mid 2^{n+1} \theta^{n+1}}{m(n+1)! \max[E_n(p; -\theta, 0), E_n(p; 0, \theta)]} + 1. \tag{3.21}$$

Thus (3.18) and (3.21) imply that

$$| q_i(x) | \leqslant \frac{M(2K)^{2n+2}}{m(n+1)!} + 1.$$

Then

$$2 \max_{1 \leqslant i \leqslant n+2} \| q_i \|_\theta \leqslant \frac{2M(2K)^{2n+2}}{m(n+1)!} + 2. \tag{3.22}$$

Since the right-hand side of (3.22) is independent of θ and the Chebyshev alternation E_θ, we see that (3.22) and (3.4) establish that

$$\lambda_f(\theta) \leqslant \frac{2M(2K)^{2n+2}}{m(n+1)!} + 2, \tag{3.23}$$

for every $\theta \leqslant \delta$. Finally, λ_δ may be taken as the right-hand side of (3.23).

If $f^{(n+1)}(x) \geqslant 0$ for all $x \in [-\delta, \delta]$, inequality (3.5) holds for all $x \in [-\delta, \delta]$ and the proof proceeds as above. ∎

EXAMPLE 2. Let $f(x) = | x |^3$, $I = [-1, 1]$, and suppose that the set of approximates is π_1. Then $f''(x) = 6 | x |$, and the polynomial of Theorem

1 can be taken as $p(x) = 6x$. Then Theorem 1 guarantees that (3.2) is a bounded set with $\delta = 1$. Thus, if the approximating class is π_1, then $f(x) = |x|^3$ has a uniform Lipschitz constant on $[-\theta, \theta]$ for every $\theta \leq 1$.

COROLLARY 1. *Let $f \in C^{n+r}[-1, 1]$, $r \geq 1$, and suppose that $f^{(n+j)}(0) = 0$, $1 \leq j \leq r-1$, but that $f^{(n+r)}(0) \neq 0$. Then there exists a $\delta > 0$ for which (3.2) is a bounded set.* (If $r = 1$ then we simply assume $f^{(n+1)}(0) \neq 0$.)

Proof. Expand $f^{(n+1)}$ in a Taylor series with remainder about the origin. Since $f^{(n+j)}(0) = 0$, $j = 1, \dots, r-1$,

$$f^{(n+1)}(x) = f^{(n+r)}(\xi) \frac{x^{r-1}}{(r-1)!}, \qquad \xi \in [-|x|, |x|].$$

Without loss of generality, we can assume that $f^{(n+r)}(x) > 0$ for all $x \in [-\delta, \delta]$, for some $\delta > 0$. Let $M \geq |f^{(n+r)}(x)| \geq m > 0$ for $x \in [-\delta, \delta]$, and define $p^{(n+1)}(x) = (x^{r-1})/(r-1)!$. Then

$$m\,|p^{(n+1)}(x)| \leq |f^{(n+1)}(x)| \leq M\,|p^{(n+1)}(x)|,$$

and the conclusion follows from Theorem 1. ∎

4. LIPSCHITZ CONSTANTS FOR ALL $\delta \leq 1$

In the previous section conditions were developed that ensure uniform Lipschitz constants on sufficiently small intervals $[-\theta, \theta] \subseteq [-1, 1]$. In this section we consider the boundedness of (1.4). Again, all approximates are from π_n unless stated otherwise.

THEOREM 2. *Suppose that $f \in C[-1, 1]$ satisfies the hypothesis of Theorem 1. Define $f_\delta(x) = f(\delta x)$ for $x \in [-1, 1]$, with δ as in Theorem 1, and assume $f_\delta \notin \pi_n$. Then (1.4) is a bounded set. That is, if the f of Theorem 1 is not a polynomial from π_n on $[-\delta, \delta]$ then (1.4) is bounded.*

Proof. Theorem 1 implies that (3.2) is bounded. The set described by

$$\Gamma = \{f_\beta(x) = f(\beta x) \mid \delta \leq \beta \leq 1\},$$

where $x \in [-1, 1]$, is a compact subset of $C[-1, 1]$. Furthermore, the hypothesis $f_\delta \notin \pi_n$ implies that $\Gamma \cap \pi_n = \varnothing$. Consequently [4, Theorem 3] implies that there is a constant λ_Γ so that (1.2) holds. Thus, since Γ is essentially the collection of restrictions of f to $[-\beta, \beta]$ for $\delta \leq \beta \leq 1$ it easily follows that

$$\{\lambda_f(\beta) \mid \delta \leq \beta \leq 1\} \tag{4.1}$$

is bounded by λ_Γ. The conclusion then follows from the boundedness of (3.2) and (4.1). ∎

COROLLARY 2. *If f is an analytic function on* $[-1, 1]$, *then* (1.4) *is a bounded set.*

Proof. If $f \in \pi_n$ on $[-1, 1]$, then the bound is 2. If $f \notin \pi_n$, the result follows from Corollary 1 and Theorem 2. ∎

If $f(\mu x) \in \pi_n$ for some μ, $0 < \mu < 1$, but $f(x) \notin \pi_n$, $-1 \leqslant x \leqslant 1$, then Theorem 2 does not guarantee that (1.4) is bounded. The following modification of Example 1 demonstrates that in this case (1.4) may be unbounded. We consider approximates from π_1 here, as in Examples 1 and 2.

EXAMPLE 3. As in Example 1, let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers satisfying $3 \leqslant n_1$ and $n_k^2 < n_{k+1}$ for $k = 1, 2,\ldots$ Choose μ such that $0 < \mu < \frac{2}{3}$. Define ϵ_k as in (2.9), $k = 1, 2,\ldots$, and let t_k, η_k, α_k, β_k, γ_k, δ_k, τ_k, and z_k be as in (2.10), $k = 1, 2,\ldots$. Let $-1 < t_k' < \sigma_k' < \eta_k' < \alpha_k' < \beta_k' < \gamma_k' < \delta_k' < \tau_k' < z_k' < 1$, $k = 1, 2,\ldots$, be as follows:

$$\begin{aligned} t_k' &= -\mu + t_k ;\\ \sigma_k' &= \mu;\\ \eta_k' &= \mu + \eta_k ;\\ \alpha_k' &= \mu + \alpha_k ;\\ \beta_k' &= \mu + \beta_k ;\\ \gamma_k' &= \gamma + \gamma_k ;\\ \delta_k' &= \mu + \delta_k ;\\ \tau_k' &= \mu + \tau_k ;\\ z_k' &= \mu + z_k . \end{aligned} \tag{4.2}$$

Let c_k' and a_k', $k = 1, 2,\ldots$, be positive constants to be determined later. We now define for each $k = 1, 2,\ldots$, $\bar{s}_k$ and $\bar{g}_k$ exactly as in (2.5) and (2.6) with c, a, t, σ, η, α, β, γ, δ, τ, and z replaced by c_k', a_k', and the corresponding entries from (4.2).

Now let

$$\bar{f}(x) = \sum_{j=1}^{\infty} \bar{s}_j(x) \tag{4.3}$$

for $-1 \leqslant x \leqslant 1$, and

$$\bar{f}_k(x) = \bar{g}_k(x) + \sum_{j=k+1}^{\infty} \bar{s}_j(x), \tag{4.4}$$

for $-1 \leqslant x \leqslant 1$ and $k = 1, 2,\ldots$.

The rest of the argument proceeds as in Example 1 with

$$a_k' = c_k' \left(\frac{\epsilon_k}{2}\right)^8 \frac{n_k}{1 - 2\mu n_k},$$

with $c_1' = 1$, and with c_k' chosen recursively to satisfy $c'_{k+1} < c_k'$ and

$$c'_{k+1} \left(\frac{\epsilon_{k+1}}{2}\right)^8 < a_k'(\eta_k' - \mu) - c_k' \left(\frac{\epsilon_k}{2}\right)^8.$$

Then on $[t_k', z_k'] = [-z_k', z_k']$ the polynomials of best approximation from π_1 to $\bar{f}$ and $\bar{f}_k$, respectively, are $\bar{p}_k(x) = 0$ and $\bar{q}_k(x) = a_k'(x - \mu)$, respectively. Also

$$\|\bar{f} - \bar{f}_k\|_{z_k'} = a_k'\delta_k$$

and

$$\|\bar{p}_k - \bar{q}_k\|_{z_k'} = a_k'(z_k - 2\mu)$$

and consequently

$$\lim_{k \to \infty} \frac{\|\bar{p}_k - \bar{q}_k\|_{z_k'}}{\|\bar{f} - \bar{f}_k\|_{z_k'}} = \infty,$$

where these assertions follow as in Example 1. As before, the c_k', $k = 1, 2, \ldots$ are chosen small enough to ensure $\bar{f} \in C^2[-1, 1]$.

Example 3 demonstrates the existence of a function $\bar{f} \in C^2[-1, 1]$ that is a polynomial of degree at most one on $[-\mu, \mu] \subseteq [-1, 1]$ (and hence (3.2) is bounded with $\lambda_\mu = 2$), that is not a polynomial of degree one on $[-\beta, \beta]$ for any $\beta > \mu$, and for which (1.4) is an unbounded set.

5. Concluding Remarks

Because of known relationships between Lipschitz and strong unicity constants, (cf. [2, p. 82; 3]), all of the above results have direct implications concerning the growth of the corresponding strong unicity constants.

Although Examples 1 and 3 have been constructed using approximates from π_1, direct modifications will provide examples from any π_n.

References

1. W. M. Bartelt, On Lipschitz conditions, strong unicity, and a theorem of A. K. Cline, *J. Approximation Theory* **14** (1975), 245–250.
2. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.

3. A. K. CLINE, Lipschitz conditions on uniform approximation operators, *J. Approximation Theory* **8** (1973), 160–172.
4. M. S. HENRY AND D. SCHMIDT, Continuity theorems for the product approximation operator, *in* "Theory of Approximation with Applications" (A. G. Law and B. N. Sahney, Ed.), pp. 24–42, Academic Press, New York, 1976.
5. M. S. HENRY AND K. L. WIGGINS, Applications of approximation theory to the initial value problem, *J. Approximation Theory* **17** (1976), 66–85.
6. G. G. LORENTZ, "Approximation of Functions," Holt, Rinehart and Winston, New York, 1966.
7. H. MAEHLY AND CH. WITZGALL, Tschebyscheff-Approximationen in kleinen intervallen I: Approximation durch polynome, *Numer. Math.* **2** (1960), 142–150.
8. G. MEINARDUS, "Approximation of Functions: Theory and Numerical Methods," Springer–Verlag, New York, 1967.
9. J. C. C. NITSCHE, Über die Abhängigkeit der Tschebyscheffschen Approximierenden einer differenzierbaren Funktion rom Intervall, *Numer. Math.* **4** (1962), 262–276.
10. S. J. POREDA, Counterexamples in best approximation, *Proc. Amer. Math. Soc.* **56** (1976), 167–171.